Proceedings of the 5th International Conference on Nonlinear Dynamics
ND-KhPI2016
September 27-30, 2016, Kharkov, Ukraine

Modeling of Slow Plasticity Waves

R.R. Labibov[1*], Yu.A Chernyakov[1]

Abstract
Quasi-static uniaxial loading of a bar with a length L is considered. Mechanical properties of a material in a point are defined by the segment of negative slope on stress-strain diagram which follows the section of elastic deformation. The deformation in specimen is uniform until the stress exceeds the peak yielding stress. The analytical solution shows that stress-strain diagram of the specimen has a yielding plateau. It is shown that the time for a slow wave to advance by a distance equal to the localized band width δ is the same as it is required for a plastic wave to run along the whole bar length.

Keywords
Yielding, hardening, strain softening, localization, Lüders bands.

[1] Department of Theoretical and Applied Mechanics, Oles Honchar Dnepropetrovsk National University, Dnepropetrovsk 49050, Ukraine
* **Corresponding author:** postrediori@gmail.com

Introduction

A large number of industrial materials such as polymers, concretes and soils exhibit strain-softening behavior under certain conditions. Stresses in these mediums tend to drop after exceeding the limit values while localizing strains progress, i.e. the stress-strain diagram has a segment of negative slope. Problems of dynamics of these phenomena are discussed in [1].

The paper considers materials which have slopes of softening as well as segments of hardening on a stress-strain diagram (e.g. materials with a yielding peak). Stress-strain curve of material according to the experimental studies [2] and static analysis is shown on the Fig. 1. Softening after the critical stress leads to localization of further deformations in narrow bands that are predecessors of material failure. Dynamical problems like the one presented in [3] are rarely studied.

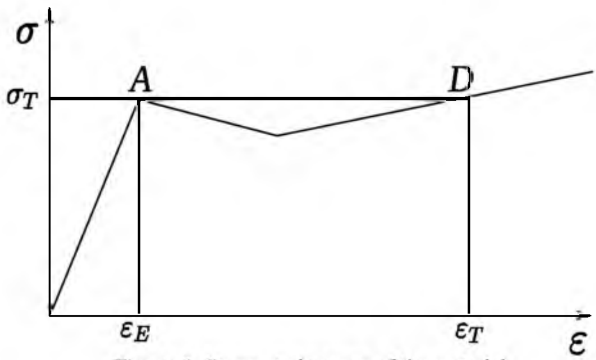

Figure 1. Stress-strain curve of the material

Direct usage of a model with softening in classical continuum usually doesn't lead to well-posed problem. The field equations that describe the motion of the body lose hyperbolicity and become elliptic as soon as strain softening occurs. In fact, domain splits into an elliptic part, where the waves have imaginary velocities and are not able to propagate (i.e. standing waves), and hyperbolic part with propagating waves.

The initial value problem can no longer be a proper description of the underlying physical problem since it becomes ill-posed. Due to the fact that standing waves are not able to propagate localization domains stay limited to a line with zero thickness (or a discrete plain in three-dimensional continuum). Spurious wave reflections occur on these localization bands with zero thickness and the energy consumed in the failure domains is zero. The finite element solution which tries to capture the localization band of zero thickness results in a mesh sensitivity [5]. A number of experiments show the instability of material's behavior. The loss of stability occurs when shifting from an elastic state (point *A* on Fig. 1) to a softening state (point *D* on Fig. 1). Lüders bands are known to appear on a specimen during this bifurcation processes.

Localization is widely discussed in theoretical and experimental studies [3, 4]. At the same time the problem of localized plastic deformation in uniform stress-strain state is not actively studied.

1. Problem of uniaxial bar tension

Uniaxial tension of a bar exhibiting both softening and hardening is considered (Fig. 2). Originally the bar is in a pre-loaded condition on a yielding limit ($\sigma=\sigma_T$). Further loading is unable to cause uniform plastic state in a specimen due to the start softening processes. Transition to plasticity is known to occur in a limited band, that is able to propagate along the bar [2] with a certain velocity V_l. Thus three limited domains in a specimen occur: a) elastic state domain with a strain $\varepsilon=\varepsilon_E$, b) plastic region with $\varepsilon=\varepsilon_T=\varepsilon_E+\varepsilon_L$, c) and a localized area with a width of δ which corresponds to the transition between elastic and plastic states. This is the reason stress-strain curve of a specimen differs from the material curve. Localization band properties such as the width δ and propagation velocity V_l are the parameters of the material and can be determined experimentally [2, 3].

Analytical solution for a stress field in the bar is obtained with defining separate equations for elastic and plastic domains and boundary conditions for the localization band. Boundaries within the medium should be considered to be able to propagate.

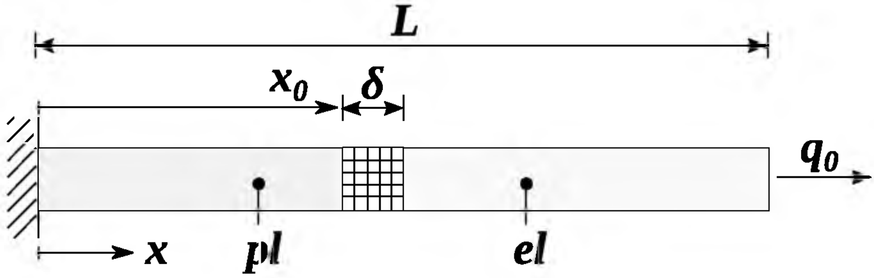

Figure 2. Uniaxial bar loading

Initial conditions:

$$\begin{aligned} u(x,0) &= 0 \\ \dot{u}(x,0) &= 0 \end{aligned} \tag{1}$$

Boundary conditions:

$$\begin{aligned} u(0,t) &= 0 \quad t \ge 0 \\ \sigma(L,t) &= 0 \quad t < 0 \\ \sigma(L,t) &= q_0 \quad t \ge 0 \end{aligned} \tag{2}$$

Mechanical conditions for elastic and plastic segments:

$$\begin{cases} \sigma = \sigma_T \\ \varepsilon = \varepsilon_E \qquad x > x_L(t) + \delta \\ u_{tt} - \frac{1}{\rho} E u_{xx} = 0 \end{cases} \qquad \begin{cases} \sigma = \sigma_T \\ \varepsilon = \varepsilon_E + \varepsilon_L \qquad x < x_L(t) \\ u_{tt} - \frac{1}{\rho} H u_{xx} = 0 \end{cases} \tag{3}$$

Plastic strain condition for a localized strip:

$$0 < \varepsilon_{loc}^{p} < \varepsilon_L \tag{4}$$

Localization strip boundary conditions:

$$\begin{aligned} &\varepsilon = \varepsilon_E \quad x = x_L(t) + \delta \\ &\varepsilon = \varepsilon_E + \varepsilon_L \quad x = x_L(t) \end{aligned} \tag{5}$$

2. Analytical solution of a uniaxial bar tension

The classical one-dimension wave equation reads:

$$\frac{1}{c_k^2} u_{tt} - u_{xx} = 0 \tag{6}$$

in which the parameters with $k=1$ stand for an elastic wave and $k=2$ stand for a plastic wave. Solutions of these equations consist of waves f_k and g_k propagating in the characteristic directions:

$$u_k(x,t) = f_k(x - c_k t) + g_k(x + c_k t) \tag{7}$$

After a change of variables one has the following:

$$u_k(x,t) = f_k\left(t - \frac{L-x}{c_k}\right) + g_k\left(t - \frac{L+x}{c_k}\right) \tag{8}$$

in which f_k are the waves propagating to the left and g_k are reflected waves. The solution for f_k for the given initial and boundary conditions reads:

$$\begin{aligned} f_k\left(t - (L-x)/c_k\right) &= \frac{h\left(t - (L-x)/c_k\right)}{\rho c_k} \frac{1}{A} \int_0^{t-(L-x)/c_k} F(\tau) d\tau = \\ &= \frac{h\left(t - (L-x)/c_k\right)}{\rho c_k} q_0\left(t - (L-x)/c_k\right) \end{aligned} \tag{9}$$

in which $h(t)$ is the Heaviside step function. A similar solution for g_k completes the solution for the displacement field:

$$u_k(x,t) = \frac{h\left(t - (L-x)/c_k\right)}{\rho c_k} q_0\left(t - (L-x)/c_k\right) - \frac{h\left(t - (L+x)/c_k\right)}{\rho c_k} q_0\left(t - (L+x)/c_k\right) \tag{10}$$

The strain field via $\varepsilon = \partial u/\partial x$ becomes:

$$\varepsilon_k(x,t) = \frac{q_0 h\left(t-(L-x)/c_k\right)}{E_k} + \frac{q_0 h\left(t-(L+x)/c_k\right)}{E_k} \tag{11}$$

in which E_k stands for elastic modulus E in elastic state solution and hardening modulus H for plastic state solution.

If yielding occurs after reflection of the wave (Fig. 3) the inelastic behavior of the localization domain δ is determined [4, 5].

Wave equation becomes elliptic, which means that interaction over finite distances is immediate. This means that the localization domain does not extend and remains infinitely small ($\delta \to 0$). We define a condition in an infinitely small localization band:

$$\varepsilon_E x < u(x,t) < (\varepsilon_E + \varepsilon_L)x \quad x_0 < x < x_0 + \delta \tag{12}$$

and for the remaining elastic and plastic parts of the bar we have:

$$u_k(x,t) = \frac{h\left(t-(L-x)/c_k\right)}{\rho c_k} q_0\left(t-(L-x)/c_k\right) + g_k\left(t-(L+x)/c_k\right) \tag{13}$$

in which the reflected waves g_k are unknown yet. The strain fields become:

$$\varepsilon_k(x,t) = \frac{q_0 h\left(t-(L-x)/c_k\right)}{E_k} - \frac{1}{c_k}\frac{dg_k}{d\left(t-(L+x)/c_k\right)} \tag{14}$$

Now the displacement and the strain fields are defined in all domains of the specimen. Interface conditions must be used to solve the system of equations. First, using (12) and (13) and assuming displacement continuity at the localization domain interface, yields and expression for strain jump over the localization band:

$$0 < [\![\varepsilon]\!] < \varepsilon_L \tag{15}$$

and the stress-strain derivative jump over the interface is:

$$0 < \left\| \frac{\partial \sigma}{\partial \varepsilon} \right\| < E - H \tag{16}$$

in which the brackets denote the difference. The stress-strain jump equation over the interface reads:

$$[\![\sigma]\!] = \rho V_l [\![\varepsilon]\!] \text{ or } \sigma_2 - \sigma_1 = \rho V_l(\varepsilon_2 - \varepsilon_1) \tag{17}$$

It is shown in the experimental data that the velocity of localization domain during yielding does not tend to zero ($V_l \neq 0$). Thus the stress jump over the interface is non-zero too. The velocity V_l in (0.17) is to be defined. Using motion equations to determine it:

$$\frac{1}{V_L}(\sigma_2 - \sigma_1) = \rho \varepsilon_L \tag{18}$$

The stress jump can be determined using empirical data for the velocity V_l. According to [2] $V_l \sim 10^{-5}$ *m/sec* for steels.

There is another way to consider the band propagation as shown in [3]. When moving along the bar elastic wave cannot naturally break the localized domain barrier with a coordinate $x_L(t)$. Instead of moving through it, the process of deflection occurs (solid points on the Fig. 3) which means the initiation of plastic waves in the plasticity domain. The deflection starts the process of stress field unification in on the boundary. The localized domain during this process undergoes yielding process and becomes a part of plasticity area. At the same time infinitesimal elastic boundary becomes the new localized band. When moving back the reflected plastic wave does not deflect (marks on the Fig. 3).

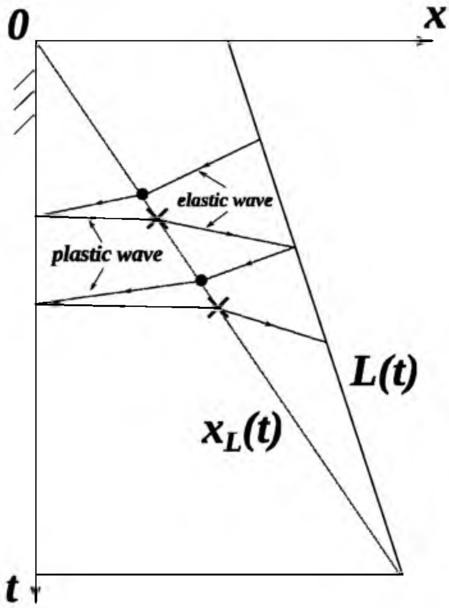

Figure 3. Localization band propagation caused by elasticity wave deflection

It takes certain time $t_0=\delta/V_l$ for the band to pass through distance δ along the bar. It should be noted that is the same time that elastic wave should pass through the boundary for two consecutive times, i.e. $t_0=2L/c_{el}$. By comparing the two equations the following formula is obtained:

$$\frac{LV_l}{\delta c_{el}}=\frac{1}{2} \tag{19}$$

This is the same equation that was determined in [3] from an experimental data.

Conclusions

Uniaxial dynamic problem solution for a material with yielding peak was determined. It is shown that a slow wave exists that divides a specimen into domains of elastic and plastic behavior. Also an estimate for the velocity of the slow wave is noted.

References

[1] Clifton R.J. Dynamic plasticity. *Journal of Applied Mechanics*, 1983, Vol. 50(4b), p. 941-952.

[2] Shaw J.A., Kyriakides S. Initiation and propagation of localized deformation in elasto-plastic strips under uniaxial tension. *International Journal of Plasticity*, 1997, Vol. 13(10), p. 837-871.

[3] Barannikova S.A., Ponomareva A.V., Zuev L.B., Vekilov Y.K. and Abrikosov I.A. Significant correlation between macroscopic and microscopic parameters for the description of localized plastic flow auto-waves in deforming alloys. *Solid State Communications*, 2012, Vol. 152(9), p. 784-787.

[4] Bažant Z.P. and Belytschko T.B. Wave propagation in a strain-softening bar: exact solution. *Journal of Engineering Mechanics*, 1985, Vol. 111(3), p. 381-389.

[5] Sluys L.J. *Wave Propagation, Localization and Dispersion in Softening Solids*. Delft: TU Delft, University of Technology; 1992.